

Presentations for rook partition monoids and algebras and their singular ideals

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Abstract

We obtain several presentations by generators and relations for the rook partition monoids and algebras, as well as their singular ideals. Among other results, we also calculate the minimal sizes of generating sets (some of our presentations use such minimal-size generating sets), and show that the singular part of the rook partition monoid is generated by its idempotents.

Keywords: Partition algebra, Partition monoid, Rook partition algebra, Rook partition monoid, Singular ideal, Presentations, Rank, Idempotent rank.

MSC: 20M05; 20M20.

1 Introduction

Diagram algebras have diverse origins and applications. For example, see Brauer [8] on invariant theory; Jones [42] and Kauffman [44] on knot theory; Temperley and Lieb [73], Jones [43] and Martin [58] on statistical mechanics; and more. One unifying theme is that diagram algebras often occur as *centraliser algebras* of classical groups, leading to various interesting extensions of (classical) *Schur-Weyl duality* [74]. For example, the *Brauer algebras* are related in this way to the orthogonal groups [8], and the *partition algebras* to symmetric groups [43, 58]. In a highly influential 2005 paper, Halverson and Ram [37] gave (among many other things) an account of Schur-Weyl duality in the case of the partition algebras, making crucial use of a *tower of algebras* that saw the partition algebras $\mathbb{C}A_k(n)$ embedded into $\mathbb{C}A_{k+1}(n)$ via an intermediate subalgebra:

$$\dots \hookrightarrow \mathbb{C}A_k(n) \hookrightarrow \mathbb{C}A_{k+\frac{1}{2}}(n) \subseteq \mathbb{C}A_{k+1}(n) \hookrightarrow \dots$$

Halverson and Ram [37] attributed their understanding of the “existence and importance” of the algebras $\mathbb{C}A_{k+\frac{1}{2}}(n)$ to Cheryl Grood, who studied them in their own right in [33], where they were called *rook partition algebras*, and given their own diagrammatic interpretation (see Section 2.1 below for details); Grood also noted that these intermediate algebras were used in earlier work of Martin [57, 59]. The reason for the name is due to a connection with the so-called *rook monoids* (and associated algebras and deformations) studied by Halverson, Solomon and others [12, 18, 32, 34, 36, 68, 72]. As noted by Grood [33], Solomon’s discovery [72] of a Schur-Weyl duality for rook monoid algebras (see also [50]) led to the investigation of a number of other “rook diagram algebras”, such the *rook Brauer algebras* [35, 60], *Motzkin algebras* [5] and more. Such studies, and other considerations often to do with representation theory and/or statistical mechanics, have led to the discovery and investigation of a great many other families of diagram algebras [6, 10, 11, 45, 61, 62, 67, 69].

In this article, we continue the study of the rook partition algebras, with our goal being to derive *presentations* by generators and relations. This continues a theme initiated by the current author in [20], where such presentations were obtained for the partition algebras themselves. Presentations are extremely useful tools for algebraists: for one thing, they allow representations (homomorphisms into other algebraic structures) to be defined by specifying the images of the generators and checking that the relations are preserved; this is especially helpful when the algebra in question is as complicated as the partition algebra. With respect to the partition algebras in particular, we refer to the recent work of Enyang on Jucys-Murphy elements [25] and seminormal forms [26], in which the presentations from [20, 37] played a crucial role. Presentations also feature heavily in the work of Lehrer and Zhang on diagram categories and invariant theory [53, 54].

Key to the approach used in [20] was the observation of Wilcox [75] (also implicit in [37]) that diagram algebras (including the partition algebras) arise as *twisted semigroup algebras* of corresponding *diagram semigroups*. This allows one to obtain information (concerning cellularity [17, 24, 31, 75] or presentations [20, 21], for example) about the algebras from corresponding information about the associated semigroups. Conversely, the theory of diagram algebras has led to a number of important families of semigroups and monoids that have been studied with increasing vigour in recent years; see for example [7, 13–17, 20, 21, 23, 24, 28, 49, 51, 56, 63, 64, 71], and especially the work of Auinger and his collaborators [1–4] on equational theories of involution semigroups. A number of other authors have studied presentations of diagram semigroups [7, 49, 56, 71]. We hope that the techniques we introduce here to deal with the more complicated *rook* partition algebras will prove useful in other investigations: for example, in the context of *quasi-partition algebras* [11] or *coloured partition algebras* [6, 69] and their rook versions [46].

As stated above, our focus in this article is on the rook partition monoid and algebra, \mathcal{RP}_n and $F^\tau[\mathcal{RP}_n]$, and their singular ideals, $\mathcal{RP}_n \setminus \mathcal{S}_n$ and $F^\tau[\mathcal{RP}_n \setminus \mathcal{S}_n]$; see Sections 2 and 5 for precise definitions. In particular, we obtain a number of presentations for each of these algebraic systems (Theorems 3.14, 4.7, 4.11, 4.12 and 5.1); among other results, we calculate the smallest sizes of generating sets (Theorems 3.15 and 4.14), and show that $\mathcal{RP}_n \setminus \mathcal{S}_n$ is generated by its idempotents (Proposition 3.1 and Remark 3.2). Our approach is quite different to previous studies of (singular) diagram semigroups and algebras [7, 20, 21, 49, 56, 71], in the sense that we first obtain a presentation for the singular rook partition monoid $\mathcal{RP}_n \setminus \mathcal{S}_n$ (the subject of Section 3), and then use this to bootstrap up to a number of presentations for the (full) rook partition monoid \mathcal{RP}_n (in Section 4). (Although this appears to be the first instance of such a “singular first” approach, we note that a somewhat similar method was used in the author’s recent work on (singular) symmetric inverse semigroups in [22].) Our approach makes crucial use of the author’s presentations for the (ordinary) partition monoid \mathcal{P}_n [20] and its singular ideal $\mathcal{P}_n \setminus \mathcal{S}_n$ [21]; these are stated in Section 2, along with various definitions, notations, background information, and illustrative examples. Finally, in Section 5, we apply general results on twisted semigroup algebras from [20] to obtain (algebra) presentations for the rook partition algebra $F^\tau[\mathcal{RP}_n]$ and its singular ideal $F^\tau[\mathcal{RP}_n \setminus \mathcal{S}_n]$; see Theorem 5.1.

2 Preliminaries

In this section, we gather the preliminary material we will need in our investigations. In particular, we define the rook partition monoids, identify several key submonoids, develop the notation and parameters needed to formulate and prove our results, and review previous results on (ordinary) partition monoids that will play a role in our study. (We postpone the definition of the rook partition *algebra* until Section 5.)

2.1 The rook partition monoid

We adapt the treatment of Grood [33]. Fix a non-negative integer n , and write $[n] = \{1, \dots, n\}$ and $[n]' = \{1', \dots, n'\}$. By a *rook partition of degree n* , we mean a set partition of a subset of $[n] \cup [n]'$: i.e., a collection $\alpha = \{A_i : i \in I\}$, for some (possibly empty) indexing set I , where the A_i are pairwise disjoint non-empty subsets of $[n] \cup [n]'$; the A_i are called the *blocks* of α . We write $\text{supp}(\alpha) = \bigcup_{i \in I} A_i$, and call this the *support* of α . The elements of $([n] \cup [n]') \setminus \text{supp}(\alpha)$ are called the *rook dots* of α . We write \mathcal{RP}_n for the set of all rook partitions of degree n . For example, the rook partition

$$\alpha = \{\{1, 2, 4, 3'\}, \{5, 6, 4', 5'\}, \{7, 8, 8'\}, \{2', 6', 7'\}, \{9', 10'\}\} \in \mathcal{RP}_{10}$$

has support $\{1, 2, 4, 5, 6, 7, 8, 2', 3', 4', 5', 6', 7', 8', 9', 10'\}$, and rook dots $3, 9, 10, 1'$.

We may represent an element $\alpha \in \mathcal{RP}_n$ graphically as follows. We arrange vertices $1, \dots, n$ in a horizontal row (increasing from left to right) with vertices $1', \dots, n'$ directly below. We colour each vertex from $\text{supp}(\alpha)$ black, and each rook dot white, and add edges between black vertices in such a way that two vertices are joined by a path if and only if they belong to the same block of α . Such a graph is called a *rook diagram* of α . The rook partition $\alpha \in \mathcal{RP}_{10}$ defined at the end of the previous paragraph is pictured in Figure 1 (top left). The graphical representation of a rook partition is not unique, but, as in [33], we regard two rook diagrams as *equivalent* if they have the same rook dots and the same connected components. Generally, we identify a rook partition $\alpha \in \mathcal{RP}_n$ with any rook diagram representing it.

To describe the product in \mathcal{RP}_n , consider two rook partitions $\alpha, \beta \in \mathcal{RP}_n$. Write $[n]'' = \{1'', \dots, n''\}$. Let α_\vee be the graph obtained from α by changing the label of each lower vertex i' to i'' . Similarly, let β^\wedge be the graph obtained from β by changing the label of each upper vertex i to i'' . Consider now the graph $\Gamma(\alpha, \beta)$ on the vertex set $[n] \cup [n]' \cup [n]''$ obtained by joining α_\vee and β^\wedge together so that each lower vertex i'' of α_\vee is identified with the corresponding upper vertex i'' of β^\wedge , and colouring a vertex i'' white in $\Gamma(\alpha, \beta)$ if it is white in either α_\vee or β^\wedge . We call $\Gamma(\alpha, \beta)$ the *product graph* of α, β . We now let $\alpha\beta \in \mathcal{RP}_n$ be the (rook partition corresponding to any) rook diagram such that:

- $\alpha\beta$ has a rook dot $x \in [n] \cup [n]'$ if and only if x is connected by a (possibly empty) path to a white vertex in $\Gamma(\alpha, \beta)$, and
- two black vertices $x, y \in [n] \cup [n]'$ are connected by a path in $\alpha\beta$ if and only if they are connected by a path in $\Gamma(\alpha, \beta)$.

An example calculation (with $n = 10$) is carried out in Figure 1.

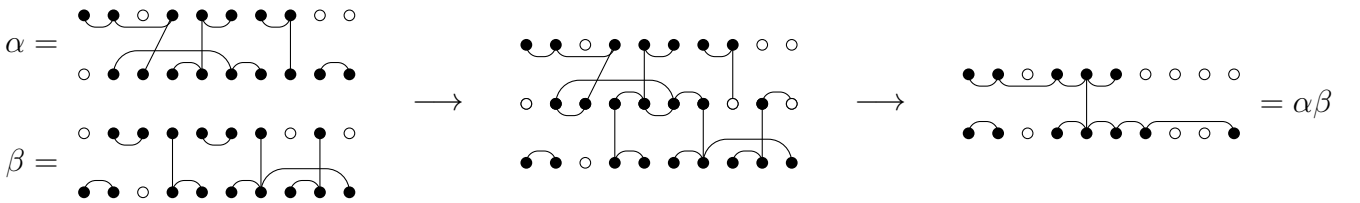


Figure 1: Two rook partitions $\alpha, \beta \in \mathcal{RP}_{10}$ (left), the product $\alpha\beta \in \mathcal{RP}_{10}$ (right), and the product graph $\Gamma(\alpha, \beta)$ (centre).

This operation is associative, and gives \mathcal{RP}_n the structure of a monoid; the identity element is

$$1 = \{\{1, 1'\}, \dots, \{n, n'\}\} = \begin{array}{c} \overset{1}{\bullet} \cdots \overset{n}{\bullet} \\ \vdots \\ \underset{1}{\bullet} \cdots \underset{n}{\bullet} \end{array} \in \mathcal{RP}_n.$$

Before we can say more about the structure of \mathcal{RP}_n , we first introduce some notation. Let $\alpha \in \mathcal{RP}_n$. We call a block of α a *transversal block* if it has non-empty intersection with both $[n]$ and $[n]'$, and

a *non-transversal block* otherwise; we will also distinguish between *upper* and *lower* non-transversal blocks (defined in an obvious way). The *rank* of α , denoted $\text{rank}(\alpha)$, is defined to be the number of transversal blocks of α . Note that $0 \leq \text{rank}(\alpha) \leq n$. For $x \in \text{supp}(\alpha)$, we write $[x]_\alpha$ for the block of α containing x ; we also define $[x]_\alpha = \{x\}$ if x is a rook dot of α . We then define the *domain* and *codomain* of α to be the sets

$$\text{dom}(\alpha) = \{x \in [n] : [x]_\alpha \cap [n]' \neq \emptyset\} \quad \text{and} \quad \text{codom}(\alpha) = \{x \in [n] : [x']_\alpha \cap [n] \neq \emptyset\}.$$

We also define the *kernel* and *cokernel* of α to be the equivalences

$$\ker(\alpha) = \{(x, y) \in [n] \times [n] : [x]_\alpha = [y]_\alpha\} \quad \text{and} \quad \text{coker}(\alpha) = \{(x, y) \in [n] \times [n] : [x']_\alpha = [y']_\alpha\}.$$

For example, with $\alpha \in \mathcal{RP}_{10}$ as in Figure 1 (top left),

$$\begin{aligned} \text{rank}(\alpha) &= 3, \quad \text{dom}(\alpha) = \{1, 2, 4, 5, 6, 7, 8\}, \quad \text{codom}(\alpha) = \{3, 4, 5, 8\}, \\ \ker(\alpha) &= (1, 2, 4 \mid 3 \mid 5, 6 \mid 7, 8 \mid 9 \mid 10), \quad \text{coker}(\alpha) = (1 \mid 2, 6, 7 \mid 3 \mid 4, 5 \mid 8 \mid 9, 10), \end{aligned}$$

using an obvious notation for equivalences.

It is immediate from the definitions that the following hold for all $\alpha, \beta \in \mathcal{RP}_n$:

$$\text{dom}(\alpha\beta) \subseteq \text{dom}(\alpha), \quad \ker(\alpha\beta) \supseteq \ker(\alpha), \quad \text{codom}(\alpha\beta) \subseteq \text{codom}(\beta), \quad \text{coker}(\alpha\beta) \supseteq \text{coker}(\beta).$$

We write $\Delta = \{(x, x) : x \in [n]\}$ for the trivial relation on $[n]$: i.e., the equality relation. The rook partition monoid \mathcal{RP}_n contains a number of important submonoids, defined as follows:

- $\mathcal{P}_n = \{\alpha \in \mathcal{RP}_n : \text{supp}(\alpha) = [n] \cup [n]'\}$, the *partition monoid* [20, 37];
- $\mathcal{I}_n = \{\alpha \in \mathcal{P}_n : \ker(\alpha) = \text{coker}(\alpha) = \Delta\}$, the *symmetric inverse monoid* [52, 55];
- $\mathcal{J}_n = \{\alpha \in \mathcal{P}_n : \text{dom}(\alpha) = \text{codom}(\alpha) = [n]\}$, the *dual symmetric inverse monoid* [29];
- $\mathcal{S}_n = \{\alpha \in \mathcal{RP}_n : \text{rank}(\alpha) = n\}$, the *symmetric group* [9]; and
- $\mathcal{R}_n = \{\alpha \in \mathcal{RP}_n : \ker(\alpha) = \text{coker}(\alpha) = \Delta, \text{supp}(\alpha) = \text{dom}(\alpha) \cup \text{codom}(\alpha)'\}$, the *rook monoid* [32, 72].

Elements of these submonoids are pictured in Figure 2, along with the various containments among the submonoids.

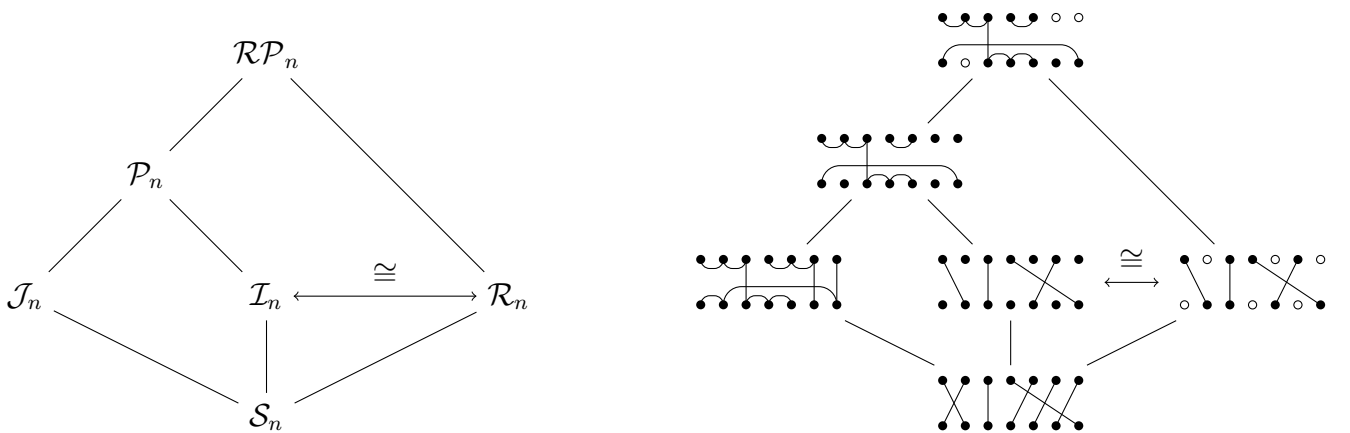


Figure 2: Important submonoids of \mathcal{RP}_n (left) and representative elements from each submonoid (right).

Note that \mathcal{I}_n and \mathcal{R}_n are isomorphic (via an obvious map $\mathcal{I}_n \rightarrow \mathcal{R}_n$ that colours all non-transversal blocks of $\alpha \in \mathcal{I}_n$ white), and that \mathcal{S}_n is the group of units of \mathcal{RP}_n (in fact, the group of units of all the

stated submonoids). It follows from [20, Theorem 32] that \mathcal{P}_n is equal to the *join* $\mathcal{I}_n \vee \mathcal{J}_n = \langle \mathcal{I}_n \cup \mathcal{J}_n \rangle$. The join $\mathcal{J}_n \vee \mathcal{R}_n$ is the *partial dual symmetric inverse monoid* studied in [47, 48]. The join $\mathcal{I}_n \vee \mathcal{R}_n$ is a “rook version” of the symmetric inverse monoid; to the author’s knowledge, it has not been explicitly studied in the literature. As noted in [33], \mathcal{RP}_n is isomorphic to the submonoid of \mathcal{P}_{n+1} consisting of all (ordinary) partitions of degree $n+1$ such that $n+1$ and $(n+1)'$ belong to the same block; these submonoids were introduced in [57, 59] and played a central role in [37].

We now describe a convenient notation for rook partitions, extending the notation introduced for ordinary partitions in [23]. With this in mind, let $\alpha \in \mathcal{RP}_n$. We write

$$\alpha = \left(\begin{array}{c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \parallel \begin{array}{c} P \\ Q \end{array} \right)$$

to indicate that α has (writing $A' = \{a' : a \in A\}$ for $A \subseteq [n]$):

- transversal blocks $A_i \cup B'_i$, for each $1 \leq i \leq r$,
- upper non-transversal blocks C_i , for each $1 \leq i \leq p$,
- lower non-transversal blocks D'_i , for each $1 \leq i \leq q$, and
- rook dots x , for each $x \in P \cup Q'$.

So, for example, with $\alpha \in \mathcal{RP}_{10}$ as in Figure 1 (top left), we have

$$\alpha = \left(\begin{array}{c|c|c|c|c} 1, 2, 4 & 5, 6 & 7, 8 & & 3, 9, 10 \\ 3 & 4, 5 & 8 & 2, 6, 7 & 9, 10 \\ & & & & 1 \end{array} \parallel \begin{array}{c} \\ \\ \end{array} \right).$$

In the above notation, it is possible that any of $r, p, q, |P|, |Q|$ could be 0, in which case we may use simplified versions of the above notation. If α has no rook dots (so that $\alpha \in \mathcal{P}_n$), we will omit P and Q , and write

$$\alpha = \left(\begin{array}{c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \right),$$

as in [23]. Similarly, if α has no rook dots and no non-transversal blocks (so that $\alpha \in \mathcal{J}_n$), then we will simply write

$$\alpha = \left(\begin{array}{c|c|c} A_1 & \cdots & A_r \\ B_1 & \cdots & B_r \end{array} \right).$$

If $\alpha \in \mathcal{I}_n$ (so $\ker(\alpha) = \text{coker}(\alpha) = \Delta$ and α has no rook dots), we will write

$$\alpha = \left(\begin{array}{c|c|c} a_1 & \cdots & a_r \\ b_1 & \cdots & b_r \end{array} \right)$$

to indicate that the transversal blocks of α are $\{a_i, b'_i\}$ (for each $1 \leq i \leq r$) and that every other block is a (non-rook) singleton. We will use other simplifications from time to time, but it will always be clear what is meant.

Finally, we note that \mathcal{RP}_n has a natural (anti-)involution $*$: $\mathcal{RP}_n \rightarrow \mathcal{RP}_n : \alpha \mapsto \alpha^*$, defined by

$$\left(\begin{array}{c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \parallel \begin{array}{c} P \\ Q \end{array} \right)^* = \left(\begin{array}{c|c|c|c|c} B_1 & \cdots & B_r & D_1 & \cdots & D_q \\ A_1 & \cdots & A_r & C_1 & \cdots & C_p \end{array} \parallel \begin{array}{c} Q \\ P \end{array} \right).$$

Diagrammatically, α^* is obtained by turning (a graphical representation of) α upside-down. This involution reflects the structure of \mathcal{RP}_n as a *regular $*$ -semigroup* (as defined in [66]). That is, the

following hold for all $\alpha, \beta \in \mathcal{RP}_n$ (as may easily be checked diagrammatically, or follows from the above-mentioned embedding $\mathcal{RP}_n \rightarrow \mathcal{P}_{n+1}$):

$$(\alpha^*)^* = \alpha, \quad (\alpha\beta)^* = \beta^*\alpha^*, \quad \alpha\alpha^*\alpha = \alpha, \quad \alpha^*\alpha\alpha^* = \alpha^*.$$

This *-regular structure leads to a duality that will help simplify several proofs, and has played a very important role in many other studies of diagram monoids; see [1–3, 20, 21, 23, 24], among many others.

We conclude this section with two structural results, concerning normal forms for \mathcal{P}_n (Proposition 2.1) and \mathcal{RP}_n (Proposition 2.3). They will both be useful on a number of occasions. The first of these gives a convenient factorisation for the elements of \mathcal{P}_n that was used in the proof of [23, Theorem 30]; its proof is easy and is omitted.

Proposition 2.1. *Let $\alpha \in \mathcal{P}_n$, and write*

$$\alpha = \left(A_1 \mid \cdots \mid A_r \mid \frac{C_1 \mid \cdots \mid C_p}{D_1 \mid \cdots \mid D_q} \right).$$

For each $i \in [r]$, choose some $a_i \in A_i$ and $b_i \in B_i$. Then $\alpha = \beta\gamma\delta$, where

$$\beta = \left(A_1 \mid \cdots \mid A_r \mid C_1 \mid \cdots \mid C_p \right), \quad \gamma = \left(a_1 \mid \cdots \mid a_r \right), \quad \delta = \left(B_1 \mid \cdots \mid B_r \mid D_1 \mid \cdots \mid D_q \right). \quad \square$$

Remark 2.2. Since $\beta, \delta \in \mathcal{J}_n$ and $\gamma \in \mathcal{I}_n$ (as defined above), Proposition 2.1 gives another proof of the above-mentioned fact that $\mathcal{P}_n = \mathcal{I}_n \vee \mathcal{J}_n = \langle \mathcal{I}_n \cup \mathcal{J}_n \rangle$.

As an example to illustrate Proposition 2.1, consider the partition

$$\alpha = \{ \{1, 2, 4, 3'\}, \{5, 6, 4', 5'\}, \{7, 8, 8'\}, \{3\}, \{9, 10\}, \{1'\}, \{2', 6', 7'\}, \{9', 10'\} \} \in \mathcal{P}_{10}.$$

Its factorisation $\alpha = \beta\gamma\delta$ (for some choice of the a_i, b_i), as in Proposition 2.1, is shown in Figure 3.

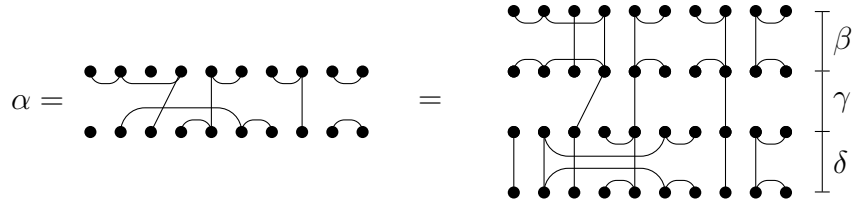


Figure 3: An illustration of the factorisation $\alpha = \beta\gamma\delta$ from Proposition 2.1, where $\alpha \in \mathcal{P}_{10}$.

Consider a subset $A \subseteq [n]$, and write $A^c = \{i_1, \dots, i_r\}$. Here and elsewhere, we will write A^c for the complement $[n] \setminus A$. We define

$$\bar{\sigma}_A = \left(i_1 \mid \cdots \mid i_r \mid \frac{A}{A} \right)$$

to be the (unique) element of \mathcal{RP}_n with rook dots x for each $x \in A \cup A'$, and transversal blocks $\{j, j'\}$ for each $j \in A^c$. (The reason for the (over-line) notation will become clear shortly.) So, for example, with $n = 10$ and $A = \{2, 4, 7, 8, 10\}$,

$$\bar{\sigma}_A = \begin{array}{ccccccccc} \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array} \in \mathcal{RP}_{10}.$$

To simplify the statement of the next result, we will allow ourselves to refer to the empty set \emptyset as a block of any element of \mathcal{P}_n .

Proposition 2.3. *For any rook partition $\alpha \in \mathcal{RP}_n$, $\alpha = \bar{\sigma}_P \beta \bar{\sigma}_Q$ for unique (possibly empty) subsets $P, Q \subseteq [n]$ and a unique partition $\beta \in \mathcal{P}_n$ such that P and Q' are (possibly empty) blocks of β . Moreover, if $\alpha \in \mathcal{RP}_n \setminus \mathcal{S}_n$, then $\beta \in \mathcal{P}_n \setminus \mathcal{S}_n$.*

Proof. Write

$$\alpha = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ \hline B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \parallel \begin{array}{c} P \\ Q \end{array} \right).$$

Let $\beta \in \mathcal{P}_n$ be the partition obtained by replacing the rook dots from α with the (possibly empty) blocks P and Q' . It is then clear that $\alpha = \bar{\sigma}_P \beta \bar{\sigma}_Q$. This establishes the existence part of the result. For the uniqueness part, suppose that $\alpha = \bar{\sigma}_A \gamma \bar{\sigma}_B$ where A and B' are (possibly empty) blocks of γ . The rook dots of $\bar{\sigma}_A \gamma \bar{\sigma}_B$ are clearly the elements of $A \cup B'$. The blocks of γ other than A and B' also clearly coincide with the (non-rook) blocks of $\bar{\sigma}_A \gamma \bar{\sigma}_B$. This all shows that $A = P$, $B = Q$ and $\gamma = \beta$.

Finally, suppose $\alpha \in \mathcal{RP}_n \setminus \mathcal{S}_n$. If α has a rook dot, then β has a non-transversal block. If α has no rook dots, then $\beta = \alpha$. So in either case, $\beta \in \mathcal{P}_n \setminus \mathcal{S}_n$. \square

Remark 2.4. An alternative factorisation $\alpha = \bar{\sigma}_P \beta \bar{\sigma}_Q$ could be proved, with $P \cup Q'$ a (possibly empty) block of β (instead of separate blocks P, Q'). However, the implication $\alpha \in \mathcal{RP}_n \setminus \mathcal{S}_n \Rightarrow \beta \in \mathcal{P}_n \setminus \mathcal{S}_n$ would not hold in general: namely, if $\alpha \in \mathcal{R}_n$ and $\text{rank}(\alpha) = n - 1$, then $\beta \in \mathcal{S}_n$.

As an example to illustrate Proposition 2.3, consider (once again) the rook partition

$$\alpha = \{ \{1, 2, 4, 3'\}, \{5, 6, 4', 5'\}, \{7, 8, 8'\}, \{2', 6', 7'\}, \{9', 10'\} \} \in \mathcal{RP}_{10}.$$

Its factorisation $\alpha = \bar{\sigma}_P \beta \bar{\sigma}_Q$, as in Proposition 2.3, is shown in Figure 4.

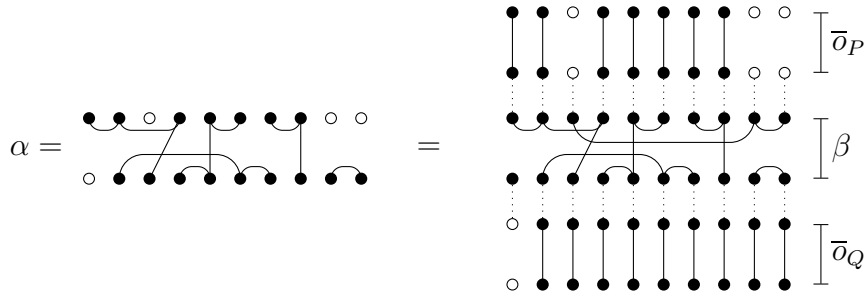


Figure 4: An illustration of the factorisation $\alpha = \bar{\sigma}_P \beta \bar{\sigma}_Q$ from Proposition 2.3, where $\alpha \in \mathcal{RP}_{10}$.

2.2 Semigroups and presentations

We will be dealing extensively with both semigroup and monoid presentations, so we now take the time to fix our notation for these, as well as some general semigroup notions. For further background on semigroups, the reader is referred to a monograph such as [38] or [41].

An equivalence relation \sim on a semigroup S is a *congruence* if $a \sim b$ and $c \sim d$ together imply $ac \sim bd$, for all $a, b, c, d \in S$. If \sim is a congruence on S , then the quotient S/\sim , which consists of all \sim -classes of S , is a semigroup under the natural induced operation. The fundamental homomorphism theorem (for semigroups) states that if $\phi : S \rightarrow T$ is a semigroup homomorphism, then $S/\ker(\phi) \cong \text{im}(\phi)$, where $\ker(\phi)$ is the congruence $\{(a, b) \in S \times S : a\phi = b\phi\}$.

Let X be an alphabet, and denote by X^+ (resp., X^*) the free semigroup (resp., free monoid) on X . If $R \subseteq X^+ \times X^+$ (resp., $R \subseteq X^* \times X^*$), we denote by R^\sharp the congruence on X^+ (resp., X^*) generated

by R . We say a semigroup (resp., monoid) S has *semigroup* (resp., *monoid*) *presentation* $\langle X : R \rangle$ if $S \cong X^+ / R^\sharp$ (resp., $S \cong X^* / R^\sharp$) or, equivalently, if there is an epimorphism $X^+ \rightarrow S$ (resp., $X^* \rightarrow S$) with kernel R^\sharp . If ϕ is such an epimorphism, we say S has *presentation* $\langle X : R \rangle$ *via* ϕ . A relation $(w_1, w_2) \in R$ will usually be displayed as an equation: $w_1 = w_2$. We will always be careful to specify whether a given presentation is a semigroup or monoid presentation.

We denote the *empty word* (over any alphabet) by 1 (so $X^* \setminus X^+ = \{1\}$ for any alphabet X). If $w = x_1 \cdots x_k$, where $x_1, \dots, x_k \in X$, we write $\ell(w) = k$ for the *length* of w . The word $x_i \cdots x_j$ is considered to be empty if either:

- (i) $i > j$ and the subscripts are understood to be increasing; or
- (ii) $i < j$ and the subscripts are understood to be decreasing.

When we are dealing with semigroup presentations, such a word will always be a subword of a larger (nonempty) word.

2.3 Presentations for $\mathcal{P}_n \setminus \mathcal{S}_n$ and \mathcal{P}_n

In Sections 3 and 4, we will give presentations for $\mathcal{RP}_n \setminus \mathcal{S}_n$ and \mathcal{RP}_n (respectively). To do this, it is crucial to know presentations for $\mathcal{P}_n \setminus \mathcal{S}_n$ and \mathcal{P}_n , and we describe such presentations (from [20, 21]) in this section. Consider alphabets $E = \{e_1, \dots, e_n\}$ and $T = \{t_{ij} : 1 \leq i < j \leq n\}$, and define a (semigroup) homomorphism

$$\phi : (E \cup T)^+ \rightarrow \mathcal{P}_n \setminus \mathcal{S}_n$$

by

$$e_i \phi = \bar{e}_i = \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} \quad \text{and} \quad t_{ij} \phi = \bar{t}_{ij} = \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad j \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}.$$

We will use symmetric notation when referring to the letters from T , so we write $t_{ij} = t_{ji}$ for all $1 \leq i < j \leq n$. Consider the relations

$$e_i^2 = e_i \quad \text{for all } i \quad (\text{R1})$$

$$e_i e_j = e_j e_i \quad \text{for distinct } i, j \quad (\text{R2})$$

$$t_{ij}^2 = t_{ij} \quad \text{for all } i, j \quad (\text{R3})$$

$$t_{ij} t_{kl} = t_{kl} t_{ij} \quad \text{for all } i, j, k, l \quad (\text{R4})$$

$$t_{ij} t_{jk} = t_{jk} t_{ki} \quad \text{for distinct } i, j, k \quad (\text{R5})$$

$$t_{ij} e_k = e_k t_{ij} \quad \text{if } k \notin \{i, j\} \quad (\text{R6})$$

$$t_{ij} e_k t_{ij} = t_{ij} \quad \text{if } k \in \{i, j\} \quad (\text{R7})$$

$$e_k t_{ij} e_k = e_k \quad \text{if } k \in \{i, j\} \quad (\text{R8})$$

$$e_k t_{ki} e_i t_{ij} e_j t_{jk} e_k = e_k t_{kj} e_j t_{ji} e_i t_{ik} e_k \quad \text{for distinct } i, j, k \quad (\text{R9})$$

$$e_k t_{ki} e_i t_{ij} e_j t_{jl} e_l t_{lk} e_k = e_k t_{kl} e_l t_{li} e_i t_{ij} e_j t_{jk} e_k \quad \text{for distinct } i, j, k, l. \quad (\text{R10})$$

The following is [21, Theorem 46].

Theorem 2.5. *The singular part of the partition monoid, $\mathcal{P}_n \setminus \mathcal{S}_n$, has semigroup presentation*

$$\langle E \cup T : (\text{R1--R10}) \rangle,$$

via ϕ . □

Now consider the alphabets $S = \{s_1, \dots, s_{n-1}\}$, $E = \{e_1, \dots, e_n\}$, $Q = \{q_1, \dots, q_{n-1}\}$, and define a (monoid) homomorphism

$$\psi : (S \cup E \cup Q)^* \rightarrow \mathcal{P}_n$$

by

$$s_i \psi = \bar{s}_i = \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}, \quad e_i \psi = \bar{e}_i = \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}, \quad q_i \psi = \bar{q}_i = \bar{t}_{i,i+1} = \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}.$$

Consider the relations

$$s_i^2 = 1 \quad \text{for all } i \quad (\text{R18})$$

$$s_i s_j = s_j s_i \quad \text{if } |i - j| > 1 \quad (\text{R19})$$

$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i - j| = 1 \quad (\text{R20})$$

$$e_i^2 = e_i \quad \text{for all } i \quad (\text{R21})$$

$$e_i e_j = e_j e_i \quad \text{for distinct } i, j \quad (\text{R22})$$

$$s_i e_j = e_j s_i \quad \text{if } j \notin \{i, i+1\} \quad (\text{R23})$$

$$s_i e_i = e_{i+1} s_i \quad \text{for all } i \quad (\text{R24})$$

$$e_i e_{i+1} s_i = e_i e_{i+1} \quad \text{for all } i \quad (\text{R25})$$

$$q_i^2 = q_i \quad \text{for all } i \quad (\text{R26})$$

$$q_i q_j = q_j q_i \quad \text{for distinct } i, j \quad (\text{R27})$$

$$s_i q_j = q_j s_i \quad \text{if } |i - j| > 1 \quad (\text{R28})$$

$$s_i s_j q_i = q_j s_i s_j \quad \text{if } |i - j| = 1 \quad (\text{R29})$$

$$q_i s_i = s_i q_i = q_i \quad \text{for all } i \quad (\text{R30})$$

$$q_i e_j = e_j q_i \quad \text{if } j \notin \{i, i+1\} \quad (\text{R31})$$

$$q_i e_j q_i = q_i \quad \text{if } j \in \{i, i+1\} \quad (\text{R32})$$

$$e_j q_i e_j = e_j \quad \text{if } j \in \{i, i+1\}. \quad (\text{R33})$$

(The jump in labels from relation (R10) to (R18) will become clear shortly.) The next result was originally stated (in a different form) in [37, Theorem 1.11], and proved in [20, Theorem 36].

Theorem 2.6. *The partition monoid, \mathcal{P}_n , has monoid presentation*

$$\langle S \cup E \cup Q : (\text{R18--R33}) \rangle$$

via ψ . □

3 The singular rook partition monoid $\mathcal{RP}_n \setminus \mathcal{S}_n$

In this section, we obtain a (semigroup) presentation for $\mathcal{RP}_n \setminus \mathcal{S}_n$, the singular part of the rook partition monoid \mathcal{RP}_n ; see Theorem 3.14. This presentation extends the presentation of $\mathcal{P}_n \setminus \mathcal{S}_n$ stated in Theorem 2.5, and uses the minimum number of generators (Theorem 3.15). We note that the method used in this section could also be used to derive a presentation for \mathcal{RP}_n itself (see Theorem 4.7) from that of \mathcal{P}_n (Theorem 2.6). However, rather than duplicating the method for \mathcal{RP}_n , we instead use Theorem 3.14 as a stepping stone towards proving Theorem 4.7 in Section 4, leading to a shorter proof. It seems that this is the first time this method (of first finding a presentation for the singular subsemigroup) has been used, though we note some similarities to the author's recent work on (singular) symmetric inverse semigroups [22].

Now define a new alphabet $O = \{o_1, \dots, o_n\}$, and for each i , let

$$\bar{o}_i = \begin{array}{c} \overset{1}{\bullet} \cdots \overset{i}{\bullet} \cdots \overset{n}{\bullet} \\ \vdots \quad \vdots \quad \vdots \\ \bullet \cdots \circ \cdots \bullet \end{array} \in \mathcal{RP}_n.$$

Note that if $A = \{i_1, \dots, i_k\} \subseteq [n]$, then \bar{o}_A (as defined in the previous section) may be factorised as $\bar{o}_A = \bar{o}_{i_1} \cdots \bar{o}_{i_k}$; in particular, $\bar{o}_i = \bar{o}_{\{i\}}$ for any $i \in [n]$. With this observation, the next result follows immediately from Proposition 2.3 and Theorem 2.5.

Proposition 3.1. *The singular part of the rook partition monoid, $\mathcal{RP}_n \setminus \mathcal{S}_n$, is generated (as a semigroup) by the set $\{\bar{e}_1, \dots, \bar{e}_n\} \cup \{\bar{t}_{ij} : 1 \leq i < j \leq n\} \cup \{\bar{o}_1, \dots, \bar{o}_n\}$. \square*

Remark 3.2. The previous result shows that $\mathcal{RP}_n \setminus \mathcal{S}_n$ is generated by its idempotents; this property is shared by many [7, 21, 24, 56], but not all [13, 14, 17], diagram monoids. Under the above-mentioned embedding $\mathcal{RP}_n \rightarrow \mathcal{P}_{n+1}$, the generators $\bar{o}_i \in \mathcal{RP}_n$ are mapped to $\bar{t}_{i,n+1} \in \mathcal{P}_{n+1}$. See also [39].

By Proposition 3.1, we may define an epimorphism

$$\Phi : (E \cup T \cup O)^+ \rightarrow \mathcal{RP}_n \setminus \mathcal{S}_n$$

by $x\Phi = \bar{x}$ for each $x \in E \cup T \cup O$. Now consider the relations

$$o_i^2 = o_i \quad \text{for all } i \quad (\text{R11})$$

$$o_i o_j = o_j o_i \quad \text{for distinct } i, j \quad (\text{R12})$$

$$o_i e_j = e_j o_i \quad \text{for distinct } i, j \quad (\text{R13})$$

$$o_i e_i o_i = o_i \quad \text{for all } i \quad (\text{R14})$$

$$e_i o_i e_i = e_i \quad \text{for all } i \quad (\text{R15})$$

$$t_{ij} o_k = o_k t_{ij} \quad \text{for any } i, j, k \quad (\text{R16})$$

$$t_{ij} o_i = t_{ij} o_j = o_i o_j \quad \text{for all } i, j. \quad (\text{R17})$$

Our aim in this section is to show that $\mathcal{RP}_n \setminus \mathcal{S}_n$ has (semigroup) presentation $\langle E \cup T \cup O : (\text{R1--R17}) \rangle$ via Φ .

Since we already know Φ is surjective, it remains to show that $\ker \Phi$ is generated by the relations (R1–R17). With this in mind, let \sim be the congruence on $(E \cup T \cup O)^+$ generated by relations (R1–R17). For $w \in (E \cup T \cup O)^+$, write $\bar{w} = w\Phi \in \mathcal{RP}_n \setminus \mathcal{S}_n$. For convenience, we will also write $\bar{1} = 1$ and $1 \sim 1$ (even though the empty word 1 does not belong to $(E \cup T \cup O)^+$).

Lemma 3.3. *We have $\sim \subseteq \ker \Phi$.*

Proof. We need to show that each of relations (R1–R17) hold as equations in $\mathcal{RP}_n \setminus \mathcal{S}_n$ when the words are replaced by their images under Φ . We already know from Theorem 2.5 that this is the case for (R1–R10). The remaining relations may easily be checked diagrammatically; we do this for (R17) in Figure 5, and leave the rest for the reader. \square

Establishing the reverse inclusion, $\ker \Phi \subseteq \sim$, forms the bulk of this section. The main step in doing this is to obtain a “word version” of Proposition 2.3; see Proposition 3.13. Our first aim is to show that any word over $(E \cup T \cup O)^+$ is \sim -equivalent to an element of $O^*(E \cup T)^*O^*$; see Lemma 3.5, the proof of which requires the next technical result.

Lemma 3.4. *Let $w \in (E \cup T)^*$ and $i \in [n]$. Then $wo_i e_i \sim w_1 w_2 w_3$ for some $w_1, w_3 \in O^*$ and $w_2 \in (E \cup T)^*$.*

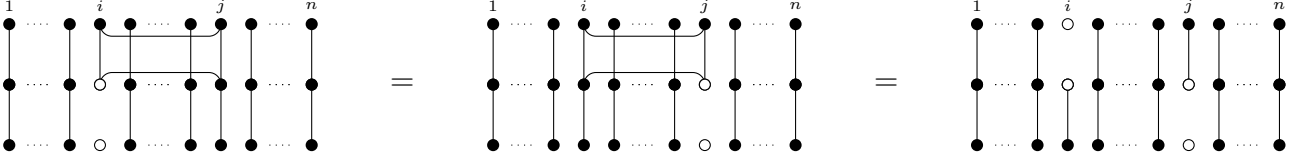


Figure 5: Diagrammatic proof of relation (R17): $\bar{t}_{ij}\bar{o}_i = \bar{t}_{ij}\bar{o}_j = \bar{o}_i\bar{o}_j$.

Proof. We prove the result by induction on $\ell(w)$, the length of w . If $\ell(w) = 0$, then we are done (with $w_1 = o_i$, $w_2 = e_i$ and $w_3 = 1$), so suppose $\ell(w) \geq 1$, and write $w = ux$, where $x \in E \cup T$ (so $u \in (E \cup T)^*$ and $\ell(u) = \ell(w) - 1$). We now consider separate cases, according to whether x belongs to E or T .

Case 1. First suppose $x = e_j \in E$. If $j = i$, then $wo_ie_i = ue_io_ie_i \sim ue_i = w$, by (R15), and we are done (with $w_1 = w_3 = 1$ and $w_2 = w$). If $j \neq i$, then $wo_ie_i = ue_jo_ie_i \sim uo_ie_ie_j$, by (R2) and (R13), and we are done after applying an induction hypothesis to uo_ie_i .

Case 2. Next suppose $x = t_{jk} \in T$. If $i \notin \{j, k\}$, then $wo_ie_i = ut_{jk}o_ie_i \sim uo_ie_it_{jk}$, by (R6) and (R16), and again we are done after applying an induction hypothesis. If $i \in \{j, k\}$, then, writing $\{l\} = \{j, k\} \setminus \{i\}$, we have $wo_ie_i = ut_{jk}o_ie_i \sim ut_{jk}o_ie_i \sim ut_{jk}e_i o_l = we_i o_l$, by (R17) and (R13), and we are done (with $w_1 = 1$, $w_2 = we_i$ and $w_3 = o_l$). \square

Lemma 3.5. *Let $w \in (E \cup T \cup O)^+$. Then $w \sim w_1 w_2 w_3$ for some $w_1, w_3 \in O^*$ and $w_2 \in (E \cup T)^*$.*

Proof. This is clearly true if $\ell(w) = 1$, so suppose $\ell(w) \geq 2$ and write $w = ux$, where $x \in E \cup T \cup O$. By an induction hypothesis, $u \sim u_1 u_2 u_3$ for some $u_1, u_3 \in O^*$ and $u_2 \in (E \cup T)^*$.

Case 1. If $x \in O$, then we are done, with $w_1 = u_1$, $w_2 = u_2$ and $w_3 = u_3 x$.

Case 2. If $x \in T$, then relation (R16) gives $u_3 x \sim x u_3$, and we are done, with $w_1 = u_1$, $w_2 = u_2 x$ and $w_3 = u_3$.

Case 3. Finally, suppose $x = e_i \in E$. If $\ell(u_3) = 0$, then we are done, so suppose $\ell(u_3) \geq 1$, and write $u_3 = o_{j_1} \cdots o_{j_k}$; by (R11) and (R12), we may assume that $j_1 < \cdots < j_k$. If $i \notin \{j_1, \dots, j_k\}$, then $u_3 e_i \sim e_i u_3$, by (R13), and we are done. So suppose $i \in \{j_1, \dots, j_k\}$; say, $i = j_l$. Then $u_3 e_i \sim o_i e_i v$, by (R12) and (R13), where $v = o_{j_1} \cdots o_{j_{l-1}} o_{j_{l+1}} \cdots o_{j_k}$. By Lemma 3.4, $u_2 o_i e_i \sim v_1 v_2 v_3$ for some $v_1, v_3 \in O^*$ and $v_2 \in (E \cup T)^*$. Putting all this together, we have $w \sim u_1 v_1 v_2 v_3 v$, and we are done, with $w_1 = u_1 v_1$, $w_2 = v_2$ and $w_3 = v_3 v$. \square

Next we wish to show (in Proposition 3.13) that the words w_1, w_2, w_3 from Lemma 3.5 may be chosen so that $\bar{w}_1, \bar{w}_2, \bar{w}_3$ correspond to the factorisation of $\bar{w} \in \mathcal{RP}_n \setminus \mathcal{S}_n$ as in Proposition 2.3. To achieve this goal, we need several intermediate results.

For $A = \{i_1, \dots, i_k\} \subseteq [n]$ with $i_1 < \cdots < i_k$, we define the words

$$o_A = o_{i_1} \cdots o_{i_k} \in O^* \quad \text{and} \quad t_A = t_{i_1 i_2} t_{i_2 i_3} \cdots t_{i_{k-1} i_k} \in T^*.$$

Note that $o_A = 1$ if $A = \emptyset$, while $t_A = 1$ if $|A| \leq 1$. Note also that $o_A \Phi = \bar{o}_A$, agreeing with our earlier use of this notation.

Lemma 3.6. *Let $A \subseteq [n]$ and let $i \in A$. Then $o_A \sim o_i t_A \sim t_A o_i$.*

Proof. By (R16), it suffices to show that $o_A \sim o_i t_A$, and we do this by induction on $|A|$. If $|A| = 1$, then there is nothing to show, since then $o_A = o_i$ and $t_A = 1$. So suppose $|A| \geq 2$, and write

$A = \{j_1, \dots, j_k\}$ with $j_1 < \dots < j_k$. Then $i = j_l$ for some $1 \leq l \leq k$, and we write $B = A \setminus \{i\}$. Since $k = |A| \geq 2$, we must have $l > 1$ or $l < k$ (or both). We will assume that $l < k$ (the other case is similar). Then

$$\begin{aligned}
o_A &\sim o_i o_B && \text{by (R12)} \\
&\sim o_i o_{j_{l+1}} t_B && \text{by an induction hypothesis} \\
&\sim o_i t_{ij_{l+1}} (t_{j_1 j_2} \cdots t_{j_{l-2} j_{l-1}} t_{j_{l-1} j_{l+1}} t_{j_{l+1} j_{l+2}} \cdots t_{j_{k-1} j_k}) && \text{by (R16) and (R17)} \\
&\sim o_i (t_{j_1 j_2} \cdots t_{j_{l-2} j_{l-1}} t_{ij_{l+1}} t_{j_{l-1} j_{l+1}} t_{j_{l+1} j_{l+2}} \cdots t_{j_{k-1} j_k}) && \text{by (R4)} \\
&= o_i (t_{j_1 j_2} \cdots t_{j_{l-2} j_{l-1}} t_{j_l j_{l+1}} t_{j_{l+1} j_{l+2}} \cdots t_{j_{k-1} j_k}) \\
&\sim o_i (t_{j_1 j_2} \cdots t_{j_{l-2} j_{l-1}} t_{j_{l-1} j_l} t_{j_l j_{l+1}} t_{j_{l+1} j_{l+2}} \cdots t_{j_{k-1} j_k}) && \text{by (R5)} \\
&= o_i t_A.
\end{aligned}$$

□

Using the previous result, we may now strengthen Lemma 3.5.

Corollary 3.7. *Let $w \in (E \cup T \cup O)^+$. Then $w \sim w_1 w_2 w_3$ for some $w_1, w_3 \in O^*$ and $w_2 \in (E \cup T)^*$, with $\ell(w_1), \ell(w_3) \leq 1$.*

Proof. By Lemma 3.5, $w \sim u_1 u_2 u_3$ for some $u_1, u_3 \in O^*$ and $u_2 \in (E \cup T)^*$. Then $u_1 \sim o_A$ and $u_3 \sim o_B$ for some (possibly empty) subsets $A, B \subseteq [n]$, by (R11) and (R12). If $A \neq \emptyset$, then choose some $i \in A$; if $B \neq \emptyset$, then choose some $j \in B$. Then

$$w \sim u_1 u_2 u_3 \sim o_A u_2 o_B \sim \begin{cases} u_2 & \text{if } A = \emptyset = B \\ o_i(t_A u_2) & \text{if } A \neq \emptyset = B \\ (u_2 t_B) o_j & \text{if } A = \emptyset \neq B \\ o_i(t_A u_2 t_B) o_j & \text{if } A \neq \emptyset \neq B, \end{cases}$$

by Lemma 3.6, and the proof is complete. □

Next we wish to show how the relations may be used to move a generator from O through a word from $(E \cup T)^*$ in certain circumstances. To do this, we require a result from [22] on (singular) symmetric inverse semigroups.

For distinct $i, j \in [n]$, we define the word $e_{ij} = e_i t_{ij} e_j \in (E \cup T)^+$, recalling that we are using symmetric notation for $t_{ij} = t_{ji}$. One easily checks that

$$\bar{e}_{ij} = \begin{cases} \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad j \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} & \text{if } i < j \\ \begin{array}{c} 1 \quad \dots \quad j \quad \dots \quad i \quad \dots \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} & \text{if } j < i. \end{cases}$$

In particular, $\bar{e}_{ij} \neq \bar{e}_{ji}$. The next result is [22, Proposition 2.2].

Proposition 3.8. *The singular part of the symmetric inverse monoid, $\mathcal{I}_n \setminus \mathcal{S}_n$, is generated (as a semigroup) by the set $\{\bar{e}_{ij} : i, j \in [n], i \neq j\}$. □*

Remark 3.9. Defining relations were also given in [22, Theorem 2.1] but we do not need those here.

Lemma 3.10. *Let $i, j \in [n]$ with $i \neq j$, and let $k \in \{i\}^c$. Then*

$$o_k e_{ij} \sim \begin{cases} e_{ij} o_k & \text{if } k \neq j \\ e_{ij} o_i & \text{if } k = j. \end{cases}$$

Proof. The case in which $k \neq j$ follows immediately from (R13) and (R16). We also have

$$o_j e_{ij} = o_j e_i t_{ij} e_j \sim e_i o_j t_{ij} e_j \sim e_i t_{ij} o_j e_j \sim e_i t_{ij} o_i e_j \sim e_i t_{ij} e_j o_i = e_{ij} o_i,$$

by (R13), (R16), (R17), (R13), respectively. \square

As usual, for $\alpha \in \mathcal{I}_n$ and $i \in \text{dom}(\alpha)$, we write $i\alpha$ for the unique element of $\text{codom}(\alpha)$ such that $\{i, (i\alpha)'\}$ is a block of α . Note that Lemma 3.10 says that $o_k e_{ij} \sim e_{ij} o_k \bar{e}_{ij}$ for all $k \in \{i\}^c = \text{dom}(\bar{e}_{ij})$; compare with the above illustration(s) of \bar{e}_{ij} .

Corollary 3.11. *Let $w \in (E \cup T)^*$ be such that $\bar{w} \in \mathcal{I}_n$. Then $o_i w \sim w o_{i\bar{w}}$ for any $i \in \text{dom}(\bar{w})$.*

Proof. The result is trivial if $\ell(w) = 0$, so suppose $\ell(w) \geq 1$. In particular, $\bar{w} \in \mathcal{I}_n \setminus \mathcal{S}_n$, so $\bar{w} = \bar{e}_{i_1 j_1} \cdots \bar{e}_{i_k j_k}$ for some i_t, j_t , by Proposition 3.8. It follows from Theorem 2.5 that $w \sim e_{i_1 j_1} \cdots e_{i_k j_k}$, and the result now follows from Lemma 3.10 and a simple induction on k . \square

For the proof of the next result, note that if

$$\alpha = \left(\begin{array}{c|c|c} A_1 & \cdots & A_r \\ \hline A_1 & \cdots & A_r \end{array} \right) \in \mathcal{J}_n,$$

then $\alpha = \overline{t_{A_1} \cdots t_{A_r}}$, where the words t_A were defined before Lemma 3.6.

Lemma 3.12. *Suppose $i, j \in [n]$ and $w \in (E \cup T)^*$ are such that i and j' belong to the same block $A \cup B'$ of \bar{w} . Then $o_i w \sim w o_j \sim o_i u o_j$ for some $u \in (E \cup T)^+$ such that $\bar{u} = (\bar{w} \setminus \{A \cup B'\}) \cup \{A, B'\}$.*

Proof. Suppose first that $\ell(w) = 0$. It follows that $w = 1$ and $i = j$, in which case $o_i w = w o_j = o_i \sim o_i e_i o_i$, by (R14), and we are done (with $u = e_i$). For the remainder of the proof, we assume that $\ell(w) \geq 1$. In particular, $\bar{w} \in \mathcal{P}_n \setminus \mathcal{S}_n$. Write

$$\bar{w} = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ \hline B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \right),$$

and let β, γ, δ be such that $\bar{w} = \beta \gamma \delta$, as in Proposition 2.1. Without loss of generality, we may suppose that $A = A_r$, $B = B_r$, and that $a_r = i$ and $b_r = j$. Let $w_1, w_3 \in T^*$ and $w_2 \in (E \cup T)^+$ be such that $\bar{w}_1 = \beta$, $\bar{w}_2 = \gamma$ and $\bar{w}_3 = \delta$. Note that $w_2 \neq 1$ since $r = \text{rank}(\bar{w}) < n$, and that $i\bar{w}_2 = i\gamma = j$. Then

$$\begin{aligned} o_i w &\sim o_i w_1 w_2 w_3 && \text{by Theorem 2.5} \\ &\sim w_1 o_i w_2 w_3 && \text{by (R16)} \\ &\sim w_1 w_2 o_j w_3 && \text{by Corollary 3.11} \\ &\sim w_1 w_2 w_3 o_j && \text{by (R16)} \\ &\sim w o_j && \text{by Theorem 2.5.} \end{aligned}$$

We also have

$$\begin{aligned}
o_i w &\sim w_1 o_i w_2 w_3 && \text{by Theorem 2.5 and (R16), as above} \\
&\sim w_1 o_i e_i o_i w_2 w_3 && \text{by (R14)} \\
&\sim o_i w_1 e_i w_2 w_3 o_j && \text{by Corollary 3.11 and (R16), as above} \\
&= o_i u o_j,
\end{aligned}$$

where $u = w_1 e_i w_2 w_3$. Note that

$$\overline{e_i w_2} = \left(\begin{array}{c|c|c} a_1 & \cdots & a_{r-1} \\ b_1 & \cdots & b_{r-1} \end{array} \right) \quad \text{so that} \quad \overline{u} = \overline{w_1 e_i w_2 w_3} = \left(\begin{array}{c|c|c|c|c|c|c} A_1 & \cdots & A_{r-1} & A & C_1 & \cdots & C_p \\ B_1 & \cdots & B_{r-1} & B & D_1 & \cdots & D_q \end{array} \right),$$

completing the proof. \square

We are now ready to describe the promised normal forms for words over $E \cup T \cup O$. As in Proposition 2.3, for the statement of the next result, we will allow ourselves to refer to the empty set \emptyset as a block of any element of \mathcal{P}_n .

Proposition 3.13. *Let $w \in (E \cup T \cup O)^+$. Then $w \sim o_A u o_B$ for some (possibly empty) $A, B \subseteq [n]$ and $u \in (E \cup T)^+$ with A and B' (possibly empty) blocks of \overline{u} .*

Proof. By Corollary 3.7, $w \sim w_1 v w_2$ for some $w_1, w_2 \in O^*$ and $v \in (E \cup T)^*$, with $\ell(w_1), \ell(w_2) \leq 1$. If $\ell(w_1) = \ell(w_2) = 0$, we are already done (with $A = B = \emptyset$ and $u = w$), so suppose this is not the case.

Case 1. Suppose first that $\ell(w_1) = 1$ and $\ell(w_2) = 0$, so $w \sim o_i v$ for some $i \in [n]$. Write $[i]_{\overline{v}} = A \cup B'$, where $A, B \subseteq [n]$, noting that A is non-empty (since $i \in A$), but B may be empty. Since $A \cup B'$ is a block of \overline{v} , it follows that $\overline{t_A v t_B} = \overline{v}$, so Theorem 2.5 gives $v \sim t_A v t_B$ (but note that this just says $1 \sim 1$ if $\ell(v) = 0$).

Subcase 1.1. Suppose first that $B = \emptyset$. Then by Lemma 3.6 and the above calculations, we have $w \sim o_i v \sim o_i t_A v \sim o_A v$, so the proof is complete in this case (with $u = v$).

Subcase 1.2. Now suppose $B \neq \emptyset$, and choose some $j \in B$. By Lemma 3.12, $w \sim o_i v \sim o_i u o_j$ for some $u \in (E \cup T)^+$ such that $\overline{u} = (\overline{v} \setminus \{A \cup B'\}) \cup \{A, B'\}$. Since A and B' are blocks of \overline{u} , it again follows from Theorem 2.5 that $u \sim t_A u t_B$. Together with Lemma 3.6, we then obtain $w \sim o_i u o_j \sim o_i t_A u t_B o_j \sim o_A u o_B$, completing the proof in this case.

Case 2. The case in which $\ell(w_1) = 0$ and $\ell(w_2) = 1$ is similar to the previous case.

Case 3. Finally, suppose $\ell(w_1) = \ell(w_2) = 1$, so $w \sim o_i v o_j$ for some $i, j \in [n]$. Write $[i]_{\overline{v}} = A \cup C'$ and $[j]_{\overline{v}} = D \cup B'$, noting that A and B are non-empty, but that C and/or D may be empty. (Note that it is possible that $[i]_{\overline{v}} = [j]_{\overline{v}}$, in which case $A = D$ and $B = C$, but this is included in the case that C and D are non-empty.)

Subcase 3.1. Suppose first that $D \neq \emptyset$, and let $k \in D$. Then $v o_j \sim o_k v$, by Lemma 3.12, so

$$w \sim o_i v o_j \sim o_i o_k v \sim \begin{cases} o_i v & \text{by (R11), if } k = i \\ o_i t_{ik} v & \text{by (R17) and (R16), if } k \neq i. \end{cases}$$

In either case, this reduces to Case 1.

Subcase 3.2. In similar fashion to the previous subcase, if $C \neq \emptyset$, then we may reduce to Case 2.

Subcase 3.3. Finally, suppose $C = D = \emptyset$. Then $w \sim o_i v o_j \sim o_i t_A v t_B o_j \sim o_A v o_B$, by Theorem 2.5 and Lemma 3.6, and we are done, with $u = v$. \square

We may now prove the main result of this section.

Theorem 3.14. *The singular part of the rook partition monoid, $\mathcal{RP}_n \setminus \mathcal{S}_n$, has semigroup presentation*

$$\langle E \cup T \cup O : (\text{R1--R17}) \rangle$$

via Φ .

Proof. It remains to show that $\ker \Phi \subseteq \sim$, so suppose $w_1, w_2 \in (E \cup T \cup O)^+$ are such that $\bar{w}_1 = \bar{w}_2$. By Proposition 3.13, $w_1 \sim o_A u_1 o_B$ and $w_2 \sim o_C u_2 o_D$, for some (possibly empty) $A, B, C, D \subseteq [n]$ and $u_1, u_2 \in (E \cup T)^+$ such that A and B' are (possibly empty) blocks of \bar{u}_1 , and C and D' are (possibly empty) blocks of \bar{u}_2 . But then $\bar{o}_A \bar{u}_1 \bar{o}_B = \bar{w}_1 = \bar{w}_2 = \bar{o}_C \bar{u}_2 \bar{o}_D$. Proposition 2.3 then gives $A = C$, $B = D$ and $\bar{u}_1 = \bar{u}_2$. Theorem 2.5 gives $u_1 \sim u_2$. Putting this all together, we have $w_1 \sim o_A u_1 o_B \sim o_A u_2 o_B = o_C u_2 o_D \sim w_2$, completing the proof. \square

The next result shows that the generating set used in Theorem 3.14 has the smallest possible size among all generating sets for $\mathcal{RP}_n \setminus \mathcal{S}_n$. Recall that the *rank* a semigroup S , denoted $\text{rank}(S)$, is the smallest size of a generating set for S . If S is generated by its idempotents, then the *idempotent rank* of S , denoted $\text{idrank}(S)$, is the smallest size of a generating set consisting entirely of idempotents. For the proof of the next result, for $1 \leq i < j \leq n$, let ε_{ij} be the equivalence relation on $[n]$ with $\{i, j\}$ as its only non-trivial equivalence class.

Theorem 3.15. *We have $\text{rank}(\mathcal{RP}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{RP}_n \setminus \mathcal{S}_n) = \frac{n^2 + 3n}{2}$.* \square

Proof. We know that $(E \cup T \cup O)\Phi$ is an idempotent generating set for $\mathcal{RP}_n \setminus \mathcal{S}_n$. Since $|E \cup T \cup O| = n + \binom{n}{2} + n = \frac{n^2 + 3n}{2}$, it follows that $\text{idrank}(\mathcal{RP}_n \setminus \mathcal{S}_n) \leq \frac{n^2 + 3n}{2}$. Since also $\text{rank}(\mathcal{RP}_n \setminus \mathcal{S}_n) \leq \text{idrank}(\mathcal{RP}_n \setminus \mathcal{S}_n)$, it remains to show that any generating set for $\mathcal{RP}_n \setminus \mathcal{S}_n$ has size at least $\frac{n^2 + 3n}{2}$. With this in mind, suppose $\mathcal{RP}_n \setminus \mathcal{S}_n = \langle \Sigma \rangle$. We claim that:

- (i) for all $i \in [n]$, Σ contains an element with domain $\{i\}^c$ and the block $\{i\}$;
- (ii) for all $i \in [n]$, Σ contains an element with domain $\{i\}^c$ and the rook dot i ; and
- (iii) for all $1 \leq i < j \leq n$, Σ contains an element with domain $[n]$ and kernel ε_{ij} .

In fact, the proofs of these are all very similar, so we just prove (i). With this in mind, let $i \in [n]$, and consider an expression $\bar{e}_i = \alpha_1 \cdots \alpha_k$, where $\alpha_1, \dots, \alpha_k \in \Sigma$. Then $n - 1 = \text{rank}(\bar{e}_i) = \text{rank}(\alpha_1 \cdots \alpha_k) \leq \text{rank}(\alpha_1) \leq n - 1$, so that $\text{rank}(\alpha_1) = n - 1$. We also have $\{i\}^c = \text{dom}(\bar{e}_i) = \text{dom}(\alpha_1 \cdots \alpha_k) \subseteq \text{dom}(\alpha_1)$, so that $\text{dom}(\alpha_1)$ is either $\{i\}^c$ or $[n]$. If the latter was the case, then (together with $\text{rank}(\alpha_1) = n - 1$) this would imply that $\ker(\alpha_1) = \varepsilon_{uv}$ for some $1 \leq u < v \leq n$, and this would give $\varepsilon_{uv} = \ker(\alpha_1) \subseteq \ker(\alpha_1 \cdots \alpha_k) = \ker(\bar{e}_i) = \Delta$, a contradiction. So, in fact, $\text{dom}(\alpha_1) = \{i\}^c$. It follows that either $\{i\}$ is a block of α_1 or else i is a rook dot of α_1 . If the latter was the case, then i would also be a rook dot of $\alpha_1 \cdots \alpha_k = \bar{e}_i$, a contradiction. So, in fact, $\{i\}$ is a block of α_1 . This completes the proof of (i). Finally, we note that Σ has (at least) n elements of type (i), n of type (ii), and $\binom{n}{2}$ of type (iii), so that $|\Sigma| \geq n + n + \binom{n}{2} = \frac{n^2 + 3n}{2}$. \square

Remark 3.16. Theorem 3.15 may also be proved using the general result [24, Theorem 5.10], utilising *Green's relations* and the *regular \ast -semigroup* structure of \mathcal{RP}_n , which we have not investigated here. It would be interesting to study \mathcal{RP}_n with the machinery developed in [24], as it would allow one to study *all* the ideals of \mathcal{RP}_n (with $\mathcal{RP}_n \setminus \mathcal{S}_n$ being the largest proper ideal). In particular, the following questions seem worthy of study:

- (i) What are the ranks of the other proper ideals of \mathcal{RP}_n ?
- (ii) Are the other proper ideals of \mathcal{RP}_n generated by their idempotents? If so, what are their idempotent ranks? Are the ranks and idempotent ranks equal?
- (iii) How many minimal-size (idempotent) generating sets are there for $\mathcal{RP}_n \setminus \mathcal{S}_n$?

These questions are all answered for the proper ideals of partition monoid \mathcal{P}_n in [24], as well as for the *Brauer monoid* and *Jones monoid*. See also [40].

4 The rook partition monoid \mathcal{RP}_n

The goal of this section is to obtain a (monoid) presentation for the rook partition monoid \mathcal{RP}_n ; in fact, we obtain several such presentations (see Theorems 4.7, 4.11 and 4.12). Our approach makes crucial use of Theorem 3.14. Specifically, we first postulate a presentation for \mathcal{RP}_n (built up from the presentation of \mathcal{P}_n in Theorem 2.6), and then show that the presentation for $\mathcal{RP}_n \setminus \mathcal{S}_n$ from Theorem 3.14 can be *embedded* (in a certain sense) in the stated presentation for \mathcal{RP}_n (see Lemmas 4.4 and 4.6).

As with Proposition 3.1, the next result follows from Proposition 2.3 and Theorem 2.6.

Proposition 4.1. *The rook partition monoid, \mathcal{RP}_n , is generated (as a monoid) by the set*

$$\{\bar{s}_1, \dots, \bar{s}_{n-1}\} \cup \{\bar{e}_1, \dots, \bar{e}_n\} \cup \{\bar{q}_1, \dots, \bar{q}_{n-1}\} \cup \{\bar{o}_1, \dots, \bar{o}_n\}.$$

□

Consider the alphabets

$$S = \{s_1, \dots, s_{n-1}\}, \quad E = \{e_1, \dots, e_n\}, \quad Q = \{q_1, \dots, q_{n-1}\}, \quad O = \{o_1, \dots, o_n\},$$

as defined earlier. By Proposition 4.1, we may define a (monoid) epimorphism

$$\Psi : (S \cup E \cup Q \cup O)^* \rightarrow \mathcal{RP}_n$$

by $x\Psi = \bar{x}$ for each $x \in S \cup E \cup Q \cup O$. Consider also the relations

$$o_i^2 = o_i \quad \text{for all } i \quad (\text{R34})$$

$$o_i o_j = o_j o_i \quad \text{for distinct } i, j \quad (\text{R35})$$

$$s_i o_j = o_j s_i \quad \text{if } j \notin \{i, i+1\} \quad (\text{R36})$$

$$s_i o_i = o_{i+1} s_i \quad \text{for all } i \quad (\text{R37})$$

$$o_i o_{i+1} s_i = o_i o_{i+1} \quad \text{for all } i \quad (\text{R38})$$

$$o_i e_j = e_j o_i \quad \text{for distinct } i, j \quad (\text{R39})$$

$$o_i e_i o_i = o_i \quad \text{for all } i \quad (\text{R40})$$

$$e_i o_i e_i = e_i \quad \text{for all } i \quad (\text{R41})$$

$$q_i o_j = o_j q_i \quad \text{for all } i, j \quad (\text{R42})$$

$$q_i o_i = q_i o_{i+1} = o_i o_{i+1} \quad \text{for all } i. \quad (\text{R43})$$

Our goal in this section is to show that \mathcal{RP}_n has (monoid) presentation $\langle S \cup E \cup Q \cup O : (\text{R18--R43}) \rangle$ via Ψ .

Since we already know Ψ is surjective, it remains to show that $\ker \Psi$ is generated by the relations (R18–R43). With this in mind, let \approx denote the congruence on $(S \cup E \cup Q \cup O)^*$ generated by relations (R18–R43). Without causing confusion, we will write $\bar{w} = w\Psi$ for all $w \in (S \cup E \cup Q \cup O)^*$. As with Lemma 3.3, the next result may easily be proved diagrammatically.

Lemma 4.2. *We have $\approx \subseteq \ker \Psi$.* \square

To establish the reverse containment, $\ker \Psi \subseteq \approx$, we need to show that $\bar{u} = \bar{v} \Rightarrow u \approx v$ for all $u, v \in (S \cup E \cup Q \cup O)^*$. The next result shows that this is the case for words u, v in certain subsets of $(S \cup E \cup Q \cup O)^*$.

Proposition 4.3. *If $u, v \in (S \cup E \cup Q)^*$ or $u, v \in (S \cup O)^*$, then $\bar{u} = \bar{v} \Rightarrow u \approx v$.*

Proof. This is clearly the case for $u, v \in (S \cup E \cup Q)^*$, since (R18–R33) are defining relations for $\mathcal{P}_n = \langle (S \cup E \cup Q)\Psi \rangle$ (Theorem 2.6). Similarly, (R18–R20, R34–R38) constitute defining relations for $\mathcal{R}_n = \langle (S \cup O)\Psi \rangle$ (see [30, Theorem 4.8] and [19, Proposition 3.22]). \square

For a word $w = s_{i_1} \cdots s_{i_k} \in S^*$, we write $w^{-1} = s_{i_k} \cdots s_{i_1}$. Note that $ww^{-1} \approx w^{-1}w \approx 1$ for all $w \in S^*$, by (R18). For $1 \leq i < j \leq n$, define words $\sigma_{ij} = s_{i+1} \cdots s_{j-1}$ (which is empty if $j = i + 1$) and $\tau_{ij} = \tau_{ji} = \sigma_{ij}^{-1} q_i \sigma_{ij}$. Note that $\bar{\tau}_{ij} = \bar{t}_{ij}$, as shown in Figure 6. Note also that $\tau_{i,i+1} = q_i$ for all i .

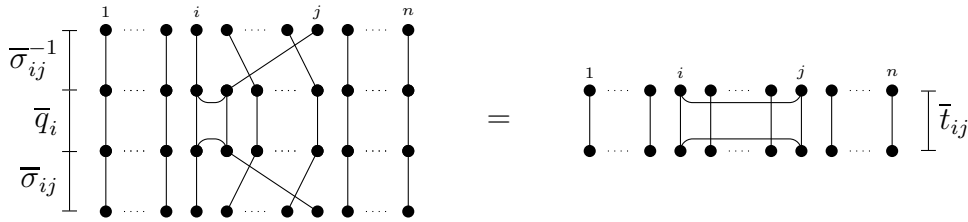


Figure 6: Diagrammatic proof that $\bar{\tau}_{ij} = \bar{t}_{ij}$.

As stated earlier, our strategy involves (somehow) linking the presentations $\langle E \cup T \cup O : (R1\text{--}R17) \rangle$ and $\langle S \cup E \cup Q \cup O : (R18\text{--}R43) \rangle$. In order to make this link explicit, we define a homomorphism

$$\rho : (E \cup T \cup O)^+ \rightarrow (S \cup E \cup Q \cup O)^*$$

by $e_i \rho = e_i$ and $o_i \rho = o_i$ (for each $1 \leq i \leq n$) and $t_{ij} \rho = \tau_{ij}$ (for each $1 \leq i < j \leq n$). It follows that $\bar{w} = \overline{w\rho}$ (i.e., $w\Phi = (w\rho)\Psi$) for all $w \in (E \cup T \cup O)^+$. Note that $\text{im}(\rho)$ is the subsemigroup of $(S \cup E \cup Q \cup O)^*$ generated by $E \cup O \cup \{\tau_{ij} : 1 \leq i < j \leq n\}$.

Recall that \sim is the congruence on $(E \cup T \cup O)^+$ generated by relations (R1–R17).

Lemma 4.4. *For any $u, v \in (E \cup T \cup O)^+$, $u \sim v \Rightarrow u\rho \approx v\rho$.*

Proof. First note that since \sim is generated by relations (R1–R17), it suffices to prove the result for each relation $u = v$ from (R1–R17). Since $w\rho \in (S \cup E \cup Q)^+$ for all $w \in (E \cup T)^+$, and since (as noted above) $\bar{w} = \overline{w\rho}$ for all $w \in (E \cup T)^+$, it follows from Proposition 4.3 that the result is true for relations (R1–R10). The result is also clearly true for relations (R11–R15) since these are precisely relations (R34), (R35) and (R39–41). It therefore remains to show that:

- (i) $\tau_{ij} o_k \approx o_k \tau_{ij}$ for distinct i, j and any k ; and
- (ii) $\tau_{ij} o_i \approx \tau_{ij} o_j \approx o_i o_j$ for distinct i, j .

Beginning with (i), let $i, j, k \in [n]$ with $i \neq j$. Put $l = k\bar{\sigma}_{ij}^{-1}$. Then $o_l \sigma_{ij} \approx \sigma_{ij} o_k$ and $\sigma_{ij}^{-1} o_l \approx o_k \sigma_{ij}^{-1}$ by Proposition 4.3 (and a simple diagrammatic check). Together with (R42), we obtain

$$\tau_{ij} o_k = \sigma_{ij}^{-1} q_i \sigma_{ij} o_k \approx \sigma_{ij}^{-1} q_i o_l \sigma_{ij} \approx \sigma_{ij}^{-1} o_l q_i \sigma_{ij} \approx o_k \sigma_{ij}^{-1} q_i \sigma_{ij} = o_k \tau_{ij}.$$

For (ii), let $i, j \in [n]$ with $i \neq j$. Then, by Proposition 4.3, (R43) and (R18), we have

$$\tau_{ij} o_i = \sigma_{ij}^{-1} q_i \sigma_{ij} o_i \approx \sigma_{ij}^{-1} q_i o_i \sigma_{ij} \approx \sigma_{ij}^{-1} o_i o_{i+1} \sigma_{ij} \approx o_i \sigma_{ij}^{-1} \sigma_{ij} o_j \approx o_i o_j.$$

A similar calculation yields $\tau_{ij} o_j \approx o_i o_j$. As noted above, this completes the proof. \square

Lemma 4.5. *Let $1 \leq i \leq n-1$ and $x \in E \cup O \cup \{\tau_{ij} : 1 \leq i < j \leq n\}$. Then $s_i x$ and $x s_i$ are \approx -equivalent to an element of $\text{im}(\rho)$.*

Proof. We consider separate cases, depending on whether x belongs to E , O or $\{\tau_{ij} : 1 \leq i < j \leq n\}$.

Case 1. Suppose first that $x \in E$. One may check diagrammatically that

$$\begin{aligned} \bar{s}_i \bar{e}_j &= \bar{e}_j \bar{s}_i = \bar{e}_j \bar{\tau}_{i+1,j} \bar{e}_{i+1} \bar{\tau}_{i,i+1} \bar{e}_i \bar{\tau}_{ij} \bar{e}_j & \text{if } j \notin \{i, i+1\} \\ \bar{s}_i \bar{e}_i &= \bar{e}_{i+1} \bar{s}_i = \bar{e}_{i+1} \bar{\tau}_{i,i+1} \bar{e}_i \\ \bar{s}_i \bar{e}_{i+1} &= \bar{e}_i \bar{s}_i = \bar{e}_i \bar{\tau}_{i,i+1} \bar{e}_{i+1}. \end{aligned}$$

It follows by Proposition 4.3 that

$$\begin{aligned} s_i e_j &\approx e_j s_i \approx e_j \tau_{i+1,j} e_{i+1} \tau_{i,i+1} e_i \tau_{ij} e_j & \text{if } j \notin \{i, i+1\} \\ s_i e_i &\approx e_{i+1} s_i \approx e_{i+1} \tau_{i,i+1} e_i \\ s_i e_{i+1} &\approx e_i s_i \approx e_i \tau_{i,i+1} e_{i+1}, \end{aligned}$$

completing the proof in this case.

Case 2. Next suppose $x = o_j \in O$, and put $k = j \bar{s}_i$ (so also $j = k \bar{s}_i$). Then $o_j s_i \approx s_i o_k$ and $s_i o_j \approx o_k s_i$, by Proposition 4.3. Together with (R40), we deduce that

$$s_i o_j \approx s_i o_j e_j o_j \approx o_k (s_i e_j) o_j \quad \text{and} \quad o_j s_i \approx o_j e_j o_j s_i \approx o_j (e_j s_i) o_k.$$

By the previous case, $s_i e_j$ and $e_j s_i$ are both \approx -equivalent to an element of $\text{im}(\rho)$, so the proof is complete in this case also.

Case 3. Finally, suppose $x = \tau_{jk}$ for some $1 \leq j < k \leq n$, and put $u = j \bar{s}_i$ and $v = k \bar{s}_i$. By Proposition 4.3, we have $\tau_{jk} s_i \approx s_i \tau_{uv}$ and $\tau_{jk} \approx \tau_{jk} e_j \tau_{jk}$. It then follows that

$$\tau_{jk} s_i \approx \tau_{jk} e_j \tau_{jk} s_i \approx \tau_{jk} (e_j s_i) \tau_{uv},$$

and, again, it follows from Case 1 that $\tau_{jk} s_i$ is \approx -equivalent to an element of $\text{im}(\rho)$. A similar calculation shows that this is the case also for $s_i \tau_{jk}$. \square

Lemma 4.6. *Let $w \in (S \cup E \cup Q \cup O)^* \setminus S^*$. Then w is \approx -equivalent to an element of $\text{im}(\rho)$.*

Proof. Put $\Sigma = E \cup O \cup \{\tau_{ij} : 1 \leq i < j \leq n\}$, noting that $\text{im}(\rho) = \langle \Sigma \rangle$ is the subsemigroup of $(S \cup E \cup Q \cup O)^*$ generated by Σ . Since $Q \subseteq \Sigma$ (as $q_i = \tau_{i,i+1}$), it suffices to show that every element of $\langle \Sigma \cup S \rangle \setminus S^*$ is \approx -equivalent to an element of $\text{im}(\rho)$. With this in mind, let $w \in \langle \Sigma \cup S \rangle \setminus S^*$, and write $w = x_1 \cdots x_k$, where $x_1, \dots, x_k \in \Sigma \cup S$. Denote by l the number of factors x_i that belong to S . We proceed by induction on l . If $l = 0$, then we already have $w \in \langle \Sigma \rangle = \text{im}(\rho)$, so suppose $l \geq 1$. Since $w \notin S^*$, there exists $1 \leq i \leq k-1$ such that either (i) $x_i \in S$ and $x_{i+1} \in \Sigma$, or (ii) $x_i \in \Sigma$ and $x_{i+1} \in S$. In either case, Lemma 4.5 tells us that $x_i x_{i+1} \approx u$ for some $u \in \text{im}(\rho) = \langle \Sigma \rangle$. But then $w \approx (x_1 \cdots x_{i-1}) u (x_{i+2} \cdots x_k)$, and we are done, after applying an induction hypothesis (noting that $(x_1 \cdots x_{i-1}) u (x_{i+2} \cdots x_k)$ has $l-1$ factors from S). \square

With these preliminary results in place, we may now prove the first of the main results of this section.

Theorem 4.7. *The rook partition monoid, \mathcal{RP}_n , has monoid presentation*

$$\langle S \cup E \cup Q \cup O : (\text{R18--R43}) \rangle$$

via Ψ .

Proof. It remains to show that $\ker \Psi \subseteq \approx$, so suppose $w_1, w_2 \in (S \cup E \cup Q \cup O)^*$ are such that $\overline{w}_1 = \overline{w}_2$. If $\overline{w}_1 \in \mathcal{S}_n$, then $w_1, w_2 \in S^*$, and $w_1 \approx w_2$, using only relations (R18–R20). For the remainder of the proof, suppose $\overline{w}_1 \notin \mathcal{S}_n$. It follows that $w_1, w_2 \in (S \cup E \cup Q \cup O)^* \setminus S^*$. So, by Lemma 4.6, $w_1 \approx u_1\rho$ and $w_2 \approx u_2\rho$ for some $u_1, u_2 \in (E \cup T \cup O)^+$. We then have

$$\overline{u}_1 = \overline{u_1\rho} = \overline{w}_1 = \overline{w}_2 = \overline{u_2\rho} = \overline{u}_2,$$

so that $u_1 \sim u_2$, by Theorem 3.14. Lemma 4.4 then gives $u_1\rho \approx u_2\rho$, so that $w_1 \approx w_2$. \square

4.1 A presentation for \mathcal{RP}_n on $n + 2$ generators

The presentation from Theorem 4.7 uses $4n - 2$ generators. In this section, we use *Tietze transformations* to reduce the size of the generating set, thereby obtaining a presentation (Theorem 4.11) in terms of S and three more generators. With this in mind, we rename $e = e_1$, $q = q_1$, $o = o_1$. Define words

$$\begin{aligned} c_i &= s_1 \cdots s_{i-1}, & E_i &= c_i^{-1}ec_i, & O_i &= c_i^{-1}oc_i & \text{for each } 1 \leq i \leq n \\ d_j &= s_2 \cdots s_j s_1 \cdots s_{j-1}, & Q_j &= d_j^{-1}qd_j & & \text{for each } 1 \leq j \leq n-1. \end{aligned}$$

Note that $c_1 = d_1 = 1$, $E_1 = e$, $O_1 = o$ and $Q_1 = q$. Diagrammatically, one may check (in similar fashion to Figure 6) that

$$\overline{e}_i = \overline{E}_i, \quad \overline{o}_i = \overline{O}_i, \quad \overline{q}_j = \overline{Q}_j \quad \text{for each } i, j.$$

It follows from Theorem 4.7 that

$$e_i \approx E_i, \quad o_i \approx O_i, \quad q_j \approx Q_j \quad \text{for each } i, j.$$

So we may transform the above presentation into $\langle S \cup \{e, q, o\} : (\text{R18–R43})' \rangle$, where each relation $(Rk)'$ is obtained from (Rk) by replacing each letter e_i, o_i, q_j by the words E_i, O_i, Q_j , respectively. This presentation is via the map $\xi : (S \cup \{e, q, o\})^* \rightarrow \mathcal{RP}_n$ defined to be the restriction to $(S \cup \{e, q, o\})^*$ of $\Psi : (S \cup E \cup Q \cup O)^* \rightarrow \mathcal{RP}_n$. Now consider the relations

$$s_i^2 = 1 \quad \text{for all } i \quad (\text{R44})$$

$$s_i s_j = s_j s_i \quad \text{if } |i - j| > 1 \quad (\text{R45})$$

$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i - j| = 1 \quad (\text{R46})$$

$$e^2 = e = eqe = eoe \quad (\text{R47})$$

$$q^2 = q = qeq = qs_1 = s_1 q \quad (\text{R48})$$

$$es_i = s_i e \quad \text{if } i \geq 2 \quad (\text{R49})$$

$$qs_i = s_i q \quad \text{if } i \geq 3 \quad (\text{R50})$$

$$s_1 es_1 e = es_1 es_1 = es_1 e \quad (\text{R51})$$

$$qs_2 qs_2 = s_2 qs_2 q \quad (\text{R52})$$

$$q(s_2 s_1 s_3 s_2) q(s_2 s_1 s_3 s_2) = (s_2 s_1 s_3 s_2) q(s_2 s_1 s_3 s_2) q \quad (\text{R53})$$

$$q(s_2 s_1 es_1 s_2) = (s_2 s_1 es_1 s_2) q \quad (\text{R54})$$

$$o^2 = o = oeo \quad (\text{R55})$$

$$os_i = s_i o \quad \text{if } i \geq 2 \quad (\text{R56})$$

$$os_1 os_1 = s_1 os_1 o = os_1 o = oq = qo \quad (\text{R57})$$

$$es_1 os_1 = s_1 os_1 e \quad (\text{R58})$$

$$q(s_2 s_1 os_1 s_2) = (s_2 s_1 os_1 s_2) q. \quad (\text{R59})$$

It is easy to check, diagrammatically, that each of relations (R44–R59) hold in \mathcal{RP}_n when the words are replaced by their images under ξ . So we may add them to the presentation, obtaining the presentation $\langle S \cup \{e, q, o\} : (\text{R18–R43})', (\text{R44–R59}) \rangle$. Our goal is to show that the relations (R18–R43)' may be removed; see Theorem 4.11. With this in mind, write \approx for the congruence on $(S \cup \{e, q, o\})^*$ generated by (R44–R59). For $w \in (S \cup \{e, q, o\})^*$, write $\overline{w} = w\xi$.

Proposition 4.8. *If $u, v \in (S \cup \{e, q\})^*$ or $u, v \in (S \cup \{o\})^*$, then $\overline{u} = \overline{v} \Rightarrow u \approx v$.*

Proof. The proof is similar to that of Proposition 4.3, since (R44–R54) include defining relations for $\mathcal{P}_n = \langle S\xi \cup \{\overline{e}, \overline{q}\} \rangle$ (see [20, Theorem 32]), while (R44–R45) and (R55–R57) include defining relations for $\mathcal{R}_n = \langle S\xi \cup \{\overline{o}\} \rangle$ (see [70] or [27]). \square

In particular, we may remove any of the relations $(Rk)'$ involving only words over $S \cup \{e, q\}$ or only words over $S \cup \{o\}$. In this way, we may remove relations (R18–R38)'. The next two results follow immediately from Proposition 4.8 (and simple diagrammatic checks).

Corollary 4.9. *Let $w \in S^*$ and $i \in [n]$. Then $w^{-1}E_iw \approx E_{i\overline{w}}$ and $w^{-1}O_iw \approx O_{i\overline{w}}$.* \square

Corollary 4.10. *Let $w \in S^*$ and $1 \leq i \leq n-1$, with $(i+1)\overline{w} = i\overline{w} + 1$. Then $w^{-1}Q_iw \approx Q_{i\overline{w}}$.* \square

Theorem 4.11. *The rook partition monoid, \mathcal{RP}_n , has monoid presentation*

$$\langle S \cup \{e, q, o\} : (\text{R44–R59}) \rangle$$

via ξ .

Proof. So far, we have transformed the presentation into $\langle S \cup \{e, q, o\} : (\text{R39–R43})', (\text{R44–R59}) \rangle$. It remains to show that relations (R39–R43)' may be removed.

(R39)'. We need to show that $O_iE_j \approx E_jO_i$ if $i \neq j$. Let $w \in S^*$ be such that $1\overline{w} = j$ and $2\overline{w} = i$. Then

$$O_iE_j \approx w^{-1}O_2ww^{-1}E_1w \approx w^{-1}O_2E_1w \approx w^{-1}E_1O_2w \approx w^{-1}E_1ww^{-1}O_2w \approx E_jO_i,$$

by Corollary 4.9, (R44) and (R58), the latter of which says “ $E_1O_2 = O_2E_1$ ”.

(R40)' and (R41)'. Here we have $O_iE_iO_i = c_i^{-1}oc_ic_i^{-1}ec_ic_i^{-1}oc_i \approx c_i^{-1}oeoc_i \approx c_i^{-1}oc_i = O_i$, by (R44) and (R55). An almost identical calculation (using (R47) instead of (R55)) deals with relation (R41)'.

(R42)'. We must show that $Q_iO_j \approx O_jQ_i$ for any i, j . Suppose first that $j \notin \{i, i+1\}$. Choose $w \in S^*$ such that $1\overline{w} = i$, $2\overline{w} = i+1$ and $3\overline{w} = j$. Then

$$Q_iO_j \approx w^{-1}Q_1ww^{-1}O_3w \approx w^{-1}Q_1O_3w \approx w^{-1}O_3Q_1w \approx w^{-1}O_3ww^{-1}Q_1w \approx O_jQ_i,$$

by Corollaries 4.9 and 4.10, and relations (R44) and (R59), the latter of which says “ $Q_1O_3 = O_3Q_1$ ”. Next, note that for any $u \in S^*$ with $1\overline{u} = i$ and $2\overline{u} = i+1$, $Q_iO_i \approx u^{-1}qou \approx u^{-1}oqu \approx O_iQ_i$, by Corollaries 4.9 and 4.10, and relations (R44) and (R57). A similar calculation gives $Q_iO_{i+1} \approx O_{i+1}Q_i$.

(R43)'. Again, let $u \in S^*$ be such that $1\overline{u} = i$ and $2\overline{u} = i+1$. Then $Q_iO_i \approx u^{-1}Q_1O_1u \approx u^{-1}O_1O_2u \approx O_iO_{i+1}$, by Corollaries 4.9 and 4.10, and relations (R44) and (R57), the relevant part of the latter of which says “ $O_1Q_1 = O_1O_2$ ”. Finally, using Proposition 4.8 and the previous calculation, we also have $Q_iO_{i+1} = Q_i s_i O_i s_i \approx Q_i O_i s_i \approx O_i O_{i+1} s_i \approx O_i O_{i+1}$. This completes the proof. \square

4.2 A presentation for \mathcal{RP}_n on 5 generators

We continue to reduce the presentation $\langle S \cup \{e, q, o\} : (\text{R44--R59}) \rangle$ from Theorem 4.11 in order to obtain a presentation with the minimal number of generators, making use of a 2-generator presentation for \mathcal{S}_n [65]. In fact, the generating set $S \cup \{e, q, o\}$ is already of minimal size when $n = 2$ (see Theorem 4.14 and Remark 4.15), so we will assume $n \geq 3$ for the rest of this section.

We now rename $s = s_1$, and add the new generator c , along with the relation $c = s_1 \cdots s_{n-1}$. It is easy to check, diagrammatically, that $\bar{s}_{i+1} = \bar{c}^i \bar{s} \bar{c}^{-i} = \bar{c}^i \bar{s} \bar{c}^{n-i}$ for all $2 \leq i \leq n-2$; here we write $\bar{c} = \bar{s}_1 \cdots \bar{s}_{n-1}$. It follows that we may remove the generators s_2, \dots, s_{n-1} from the presentation, and replace their every occurrence in the relations by the words $S_{i+1} = c^i s c^{n-i}$ (for each $1 \leq i \leq n-2$). (We also define $S_1 = s$.) The result of doing this to relation (Rk) will be denoted (Rk)'. So we have transformed the presentation to

$$\langle s, c, e, q, o : (\text{R44--R59})', c = S_1 \cdots S_{n-1} \rangle.$$

This presentation is via the map

$$\zeta : \{s, c, e, q, o\}^* \rightarrow \mathcal{RP}_n : x \mapsto \bar{x}.$$

Now consider the relations

$$c^n = (sc)^{n-1} = s^2 = (c^i s c^{n-i} s)^2 = 1 \quad \text{for all } 2 \leq i \leq \frac{n}{2} \quad (\text{R60})$$

$$e^2 = e = eqe = eoe = scec^{n-1}s = csc^{n-1}ecsc^{n-1} \quad (\text{R61})$$

$$q^2 = q = qeq = qs = sq = c^2 s c^{n-2} q c^2 s c^{n-2} = c^{n-1} s c s c^{n-1} q c s c^{n-1} s c \quad (\text{R62})$$

$$sese = eses = ese \quad (\text{R63})$$

$$qcqc^{n-1} = cqc^{n-1}q \quad (\text{R64})$$

$$qc^2qc^{n-2} = c^2qc^{n-2}q \quad (\text{R65})$$

$$qc^2ec^{n-2} = c^2ec^{n-2}q \quad (\text{R66})$$

$$o^2 = o = oeo = scoc^{n-1}s = csc^{n-1}ocsc^{n-1} \quad (\text{R67})$$

$$soso = osos = oso = oq = qo \quad (\text{R68})$$

$$qc^2oc^{n-2} = c^2oc^{n-2}q \quad (\text{R69})$$

$$esos = sose. \quad (\text{R70})$$

Theorem 4.12. *The rook partition monoid, \mathcal{RP}_n , has monoid presentation*

$$\langle s, c, e, q, o : (\text{R60--R70}) \rangle$$

via ζ .

Proof. Write \diamond for the congruence on $\{s, c, e, q, o\}^*$ generated by relations (R60–R70). We already know that ζ is surjective, and it is easy to check that $\diamond \subseteq \ker \zeta$, so we may add relations (R60–R70) to obtain the presentation

$$\langle s, c, e, q, o : (\text{R60--R70}), (\text{R44--R59})', c = S_1 \cdots S_{n-1} \rangle.$$

It remains to show that we can remove all relations apart from (R60–R70). Relations (R60–R66) contain defining relations for $\mathcal{P}_n = \langle \bar{s}, \bar{c}, \bar{e}, \bar{q} \rangle$ (see [20, Theorem 41]), so we may remove $c = S_1 \cdots S_{n-1}$ and all relations from (R44–R59)' with no occurrence of the letter o . This leaves (R55–R59)' and the relation $e = eoe$; the latter is part of (R61), so may be removed. Next note that (R60), (R67) and

(R68) contain defining relations for $\mathcal{R}_n = \langle \bar{s}, \bar{c}, \bar{o} \rangle$ (see [70] or [27]), so we may remove all relations from (R55–R59)' with no occurrence of the letters e, q . This leaves (R58)', (R59)', and the relations

$$o = oeo \quad \text{and} \quad oq = qo = oso.$$

These last relations are already part of (R67) and (R68), so we are only left with (R58)' and (R59)'. The former is just (R70). For (R59)', first observe that $\bar{s}_2 \bar{s}_1 \bar{o} \bar{s}_1 \bar{s}_2 = \bar{c}^2 \bar{o} \bar{c}^{n-2}$, so that $S_2 S_1 o S_1 S_2 \diamond c^2 o c^{n-2}$, by the above-mentioned fact that (R60–R70) contains defining relations for $\mathcal{R}_n = \langle \bar{s}, \bar{c}, \bar{o} \rangle$. Together with (R69), it then follows that $q S_2 S_1 o S_1 S_2 \diamond q c^2 o c^{n-2} \diamond c^2 o c^{n-2} q \diamond S_2 S_1 o S_1 S_2 q$. \square

Remark 4.13. The presentation in Theorem 4.12 includes defining relations for the symmetric group \mathcal{S}_n (i.e., relations (R60)), plus a fixed (i.e., independent of n) number of extra relations: 26 such extra relations, to be precise. We make no claim that this is the minimal number of additional relations required.

We conclude this section by showing that the generating set in the previous result has the minimal possible size.

Theorem 4.14. *For $n \geq 3$, $\text{rank}(\mathcal{RP}_n) = 5$.*

Proof. By Theorem 4.12, it suffices to show that any generating set for \mathcal{RP}_n has size at least 5. So suppose $\mathcal{RP}_n = \langle \Sigma \rangle$. Since $\mathcal{RP}_n \setminus \mathcal{S}_n$ is an ideal of \mathcal{RP}_n , it follows that $\Sigma \cap \mathcal{S}_n$ is a generating set for \mathcal{S}_n . Since $\text{rank}(\mathcal{S}_n) = 2$, it follows that $|\Sigma \cap \mathcal{S}_n| \geq 2$. It therefore remains to show that $|\Sigma \setminus \mathcal{S}_n| \geq 3$.

Consider an expression $\bar{e}_1 = \alpha_1 \cdots \alpha_k$, where $\alpha_1, \dots, \alpha_k \in \Sigma$. Let $1 \leq l \leq k$ be minimal so that $\alpha_l \notin \mathcal{S}_n$. Let $i = 1(\alpha_1 \cdots \alpha_{l-1})$. Then

$$\bar{e}_i = (\alpha_1 \cdots \alpha_{l-1})^{-1} \bar{e}_1 (\alpha_1 \cdots \alpha_{l-1}) = \alpha_l (\alpha_{l+1} \cdots \alpha_k) (\alpha_1 \cdots \alpha_{l-1}).$$

As in the proof of Theorem 3.15, it follows that α_l has domain $\{i\}^c$ and the block $\{i\}$. Similarly, it can be shown that $\Sigma \setminus \mathcal{S}_n$ contains: an element with domain $\{j\}^c$ and the rook dot j , for some $j \in [n]$; and an element with domain $[n]$ and non-trivial kernel. \square

Remark 4.15. The second paragraph of the previous proof works for $n = 2$ as well, showing that $\text{rank}(\mathcal{RP}_2) = \text{rank}(\mathcal{S}_2) + 3 = 4$. It is easy to check that $\text{rank}(\mathcal{RP}_n) = 1, 3$ for $n = 0, 1$.

5 Rook partition algebras

Recall from [75] that the *partition algebras* are *twisted semigroup algebras* of the partition monoids. As noted in the introduction, this understanding has led to a rich flow of information between the theories of diagram *semigroups* and diagram *algebras*: see for example [15–17, 20, 21, 24, 75]. In this section, we show how the results of previous sections lead to (algebra) presentations for the *rook partition algebras* and their singular ideals.

Let F be a commutative ring with 1, and let S be a semigroup. Recall that a *twisting* from S to F is a map $\tau : S \times S \rightarrow F$ satisfying $\tau(a, b)\tau(ab, c) = \tau(a, bc)\tau(b, c)$ for all $a, b, c \in S$. Given such a twisting, the *twisted semigroup algebra* of S over F with respect to τ , denoted $F^\tau[S]$, is defined to be the set of all finite formal F -linear combinations over S , with associative operation \star defined on basis elements (and then extended linearly) by $a \star b = \tau(a, b)ab$ for each $a, b \in S$ (where ab is the product

of a and b in S). When the twisting is trivial (i.e., $\tau(a, b) = 1 \in F$ for all $a, b \in S$), $F^\tau[S] = F[S]$ is just the (ordinary) semigroup algebra.

Given a fixed element $\delta \in F$, one may define a twisting from \mathcal{RP}_n to F as follows. For $\alpha, \beta \in \mathcal{RP}_n$, let $m(\alpha, \beta)$ denote the number of connected components in the product graph $\Gamma(\alpha, \beta)$ that involve only black vertices in the middle layer (i.e., double-dashed non-rook vertices). One may show that

$$m(\alpha, \beta) + m(\alpha\beta, \gamma) = m(\alpha, \beta\gamma) + m(\beta, \gamma) \quad \text{for all } \alpha, \beta, \gamma \in \mathcal{RP}_n.$$

As a result, one may then define a twisting

$$\tau : \mathcal{RP}_n \times \mathcal{RP}_n \rightarrow F \quad \text{by} \quad \tau(\alpha, \beta) = \delta^{m(\alpha, \beta)} \quad \text{for all } \alpha, \beta \in \mathcal{RP}_n,$$

and the resulting twisted semigroup algebra $F^\tau[\mathcal{RP}_n]$ is called the *rook partition algebra* [33]. (The reason that white vertices (i.e., rook dots) do not figure in the count of $m(\alpha, \beta)$ is due to the exact nature of the above-mentioned embedding of \mathcal{RP}_n in \mathcal{P}_{n+1} ; see [33] for more details.) Note also that τ restricts to a twisting from $\mathcal{RP}_n \setminus \mathcal{S}_n$ to F , so we may consider the *singular rook partition algebra*, $F^\tau[\mathcal{RP}_n \setminus \mathcal{S}_n]$.

A general result from [20] shows how a (monoid or semigroup) presentation for S leads to an (algebra) presentation for $F^\tau[S]$ in the case that the image of the twisting τ lies in $G(F)$, the group of units of F . The reader is referred to [20, Section 6] for full details. The next result follows immediately from [20, Theorem 44] and earlier results of the current paper (as stated below).

Theorem 5.1. *Suppose F is a commutative ring with identity, and let $\delta \in G(F)$ be a unit in F .*

- (i) *An algebra presentation for the rook partition algebra, $F^\tau[\mathcal{RP}_n]$, may be obtained from any of the above monoid presentations for \mathcal{RP}_n by:*
 - (a) *changing the relations $e_i^2 = e_i$ to $e_i^2 = \delta e_i$ in Theorem 4.7; or*
 - (b) *changing the relations $e^2 = e$ to $e^2 = \delta e$ in Theorem 4.11 or Theorem 4.12.*
- (ii) *An algebra presentation for the singular rook partition algebra, $F^\tau[\mathcal{RP}_n \setminus \mathcal{S}_n]$, may be obtained from the semigroup presentation for $\mathcal{RP}_n \setminus \mathcal{S}_n$ in Theorem 3.14 by changing the relations $e_i^2 = e_i$ to $e_i^2 = \delta e_i$. \square*

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